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# Non-minimal BRST terms for Yang–Mills theory

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**Abstract.** An extended BRST symmetry allows to obtain many non-minimal terms, which are very useful in the gauge-fixing procedure of the extended Hamiltonian. By structuring the generators on many levels, we construct the phase space for the implementation of a fourth-order symmetry. The corresponding extended Hamiltonian, containing all possible non-minimal terms, is presented for the case of Yang–Mills fields and the results obtained by this extension compared to those issued from standard BRST theory.

## 1. Introduction

One of the most powerful methods for the quantization of Yang–Mills theories consists of the extension of the phase space by using ghost-type fields, which are variables without any physical significance that do not appear in the asymptotical states. This procedure of quantization was used for the first time by 't Hooft [1] in order to overcome the inconsistencies of the Fadeev–Popov scheme. As a result of this enlargement of the phase space, it was noticed, for the Yang–Mills field [2] as well as for other gauge theories [3], that a global symmetry appears that incorporates the original local symmetry. The existence of this global symmetry offers the ideal frame for the quantization of the gauge theories and gives rise to the BRST quantization procedure.

The Hamiltonian approach of this procedure, which will be used in the present paper, was developed by Batalin, Fradkin and Vilkovisky [4, 5]. A pedagogical exposure of this approach and the analysis of the Yang–Mills field in this context has been offered in [6].

A few years later, Batalin *et al* [7, 8] proposed a new Hamiltonian formalism that joined the BRST and the anti-BRST symmetries in an sp(2) doublet. A cohomological interpretation of the BRST–anti-BRST theory has been given by Gregoire and Henneaux [9, 10]. Their construction is based on the idea of the duplication of constraints and of all generators of the extended phase space.

However, it was clear that the generators of the sp(2) symmetry exhaust only partially the structure of the non-minimal sector, many other possibilities remaining valid. A larger symmetry could offer many non-minimal terms, but, at the same time, could increase the difficulty of the ghost structure control. The necessity of ordering these variables therefore arises. Using the idea of multiplying the ghost variables and of spreading them out on many levels, the extension of the sp(2) BRST symmetry towards a sp(3) one has been achieved in [11, 12].

We intend now to take one more step on the way to a generalized structure of extended phase space in the BRST quantization and to analyse the possibility of defining a fourth-order symmetry. We have in view two main goals: to determine the possible non-minimal terms that could appear in this case in the extended Hamiltonian; and to verify the validity of this theory, by applying it to the case of Yang–Mills theory. The paper is organized as follows. In section 2 we present the structure of the extended phase space in the frame of the sp(4) BRST quantization and point out the form of the BRST charges and of the extended Hamiltonian for a first-rank theory. In section 3 we effectively write these quantities for Yang–Mills field. The comparison of our results to those issued from the quantization of the same theory in the context of a less-extended BRST symmetry is presented in section 4. Some conclusive remarks end the paper.

## 2. The fourth-order BRST symmetry

Let us consider a dynamical system that in the phase space $M_0 = \{q^i, p_i; i = \overline{1,n}\}$ is described by the Hamiltonian $H_0(q, p)$ and by the independent first-class constraints $\{G_\alpha(q, p) = 0, \alpha = \overline{1,m}\}$. The first-class condition implies the relations

$$[G_\alpha, G_\beta] = C^\gamma_{\alpha\beta} G_\gamma \qquad [H_0, G_\alpha] = V^\beta_\alpha G_\beta. \tag{1}$$

The Poisson brackets are defined as

$$[q^i, p_j] = \delta^i_j. \tag{2}$$

The structure functions $C^\gamma_{\alpha\beta}$ and $V^\beta_\alpha$ either depend on the canonical variables $\{q^i, p_i\}$ or are constant, as for the case of Yang–Mills theory.

Because of the constraints, not all the coordinates of $M_0$ are independent and, therefore, canonical quantization is not possible in this space. The BRST method employs an extended phase space $M$ that is obtained by adding some ghost variables to the real ones. In this extended space the theory is invariant in relation to a global symmetry, an invariance which could be expressed either through the differential operator $s^T$ or through the BRST charge $\Omega^T$. For an observable $A$ one has

$$s^T A \equiv [A, \Omega^T] = 0. \tag{3}$$

The nilpotency of $s^T$ asks for the validity of the master equation:

$$[\Omega^T, \Omega^T] = 0. \tag{4}$$

In the last two relations the Poisson brackets must be defined on the whole extended phase space $M$. The ghost spectrum which generates this space depends on the symmetry required. In this paper we are interested in a BRST symmetry that can be expressed as the sum of four anticommuting differentials (the BRST symmetry of fourth order):

$$s^T = \sum_{a=1}^{4} s_a \qquad s_a s_b + s_b s_a = 0 \qquad a, b = \overline{1,4}. \tag{5}$$

As in the standard BRST construction, the extended phase space can be split in the *ghost momenta complex* endowed with the Koszul differential and in *the ghosts' complex* on which the longitudinal derivative acts. Both of the two complexes are graduated using the following grades [11]:

- *the ghost number* (gh) is strictly positive for ghosts and negative for ghost momenta;
- *the level number* (lev) is a positive integer for ghosts and negative for ghost momenta;
- *the resolution degree* (res) is vanishing for ghosts and with value opposite to gh for momenta.

Using these graduations, each variable can be written in the form

$$A_\alpha \equiv \overset{(\mathrm{gh},\mathrm{lev})}{A_\alpha}\ . \tag{6}$$

Each differential $s_a$ is constructed by the Koszul differential $\delta_a$ and by the longitudinal derivative $d_a$:

$$s_a = \delta_a + d_a + \cdots.$$

Because of (5), we look for a Koszul differential of the form

$$\delta^T = \sum_{a=1}^{4} \delta_a;\ \delta_a\delta_b + \delta_b\delta_a = 0 \qquad a, b = \overline{1, 4}. \tag{7}$$

Let us now find the minimal structure of the extended phase space for the case of a fourth-order BRST symmetry. We start with the construction of the Koszul complex, generated by the ghost momenta. In the standard manner, we first introduce the $P_\alpha$*-momenta*:

$$G_\alpha = 0 \Rightarrow (\exists)\{P_{\alpha a};\ \delta_a P_{\alpha b} = \delta_{ab} G_\alpha;\ \alpha = \overline{1, m};\ a, b = \overline{1, 4}\}. \tag{8}$$

In order to find the non-trivial cycles that contain $P_{\alpha a}$, we multiply the relation (8), *without summation*, by the fourth-order total anti-symmetric tensor. Because of the $\delta_{ab}$ symmetry we have

$$\delta_a(\epsilon_{abcd} P_{\alpha b}) = 0 \qquad \epsilon_{1234} = 1 \qquad a, b, c, d = \overline{1, 4}. \tag{9}$$

The relations (9) ask for new generators. Let us now introduce the $\pi$*-momenta* of second order:

$$\{\pi_{\alpha ab};\ a \neq b;\ \delta_a \pi_{\alpha bc} = \epsilon_{abcd} P_{\alpha d};\ a, b, c, d = \overline{1, 4}\}. \tag{10}$$

New non-trivial cocycles are generated by (10). We obtain their expressions by multiplying (10) with $\delta_{ac}$, again without summation. We have

$$\delta_a \pi_{\alpha ab} = 0 \tag{11}$$

and we might add new generators to the ghost-momenta complex. Let us consider the set of first-order $\pi$-momenta:

$$\{\pi_{\alpha a};\ \delta_a \pi_{\alpha b} = \pi_{\alpha ab};\ a, b = \overline{1, 4}\}. \tag{12}$$

The construction of the Koszul complex is achieved by considering a final set of generators: the $\pi$-momenta of order zero:

$$\{\pi_\alpha;\ \delta_a \pi_\alpha = \pi_{\alpha a};\ a = \overline{1, 4}\}. \tag{13}$$

In conclusion, the Koszul complex suitable for the implementation of fourth-order BRST symmetry is generated by the following set of generators:

$$P_A \equiv \{p_i, P_{\alpha a}, \pi_{\alpha ab}, \pi_{\alpha a}, \pi_\alpha;\ i = \overline{1, n};\ \alpha = \overline{1, m};\ a, b = \overline{1, 4}\}. \tag{14}$$

*Remark 1.* The graduation of the variables (14) is given by:

- the ghost numbers and the resolution degrees:

$$\mathrm{gh}(p_i) = -\mathrm{res}(p_i) = 0 \qquad \mathrm{gh}(P_{\alpha a}) = -\mathrm{res}(P_{\alpha a}) = -1$$
$$\mathrm{gh}(\pi_{\alpha ab}) = -\mathrm{res}(\pi_{\alpha ab}) = -2 \qquad \mathrm{gh}(\pi_{\alpha a}) = -\mathrm{res}(\pi_{\alpha a}) = -3$$
$$\mathrm{gh}(\pi_\alpha) = -\mathrm{res}(\pi_\alpha) = -4$$

- the level numbers:

$$\mathrm{lev}(p_i) = 0 \qquad \mathrm{lev}(P_{\alpha a}) = a - 1 \qquad \mathrm{lev}(\pi_{\alpha ab}) = a + b - 8$$
$$\mathrm{lev}(\pi_{\alpha a}) = a - 7 \qquad \mathrm{lev}(\pi_\alpha) = -6.$$

*Remark 2.* One can see that there are two different second-order $\pi$-momenta, namely $\pi_{\alpha 14}$ and $\pi_{\alpha 23}$, with the same level number:

$$\mathrm{lev}(\pi_{\alpha 14}) = \mathrm{lev}(\pi_{\alpha 23}) = -3.$$

This fact can be explained by the requirement (7) imposed for $\delta$-operators. If we project on different values of the level-number the relation

$$\delta_T^2 = (\delta_1 + \delta_2 + \delta_3 + \delta_4)^2 = 0$$

we obtain that, on the $L^{(-3)}$-level, the following relation might be fulfilled:

$$(\delta_1\delta_4 + \delta_4\delta_1) + (\delta_2\delta_3 + \delta_3\delta_2) = 0. \tag{15}$$

The requirement (7) is stronger than (15) and we need two different sets of generators to ensure that each bracket vanishes separately.

*Remark 3.* The canonical structure of the extended phase space is accomplished by defining the ghost sector. It is generated by the variables

$$Q^A \equiv \{q^i, Q^{\alpha a}, \lambda^{\alpha ab}, \lambda^{\alpha a}, \lambda^\alpha; i = \overline{1, n}; \alpha = \overline{1, m}; a, b = \overline{1, 4}\}. \tag{16}$$

We now pass to the problem of finding the BRST charges, $\Omega_a$, and the extended Hamiltonian. For this, we solve the following equations:

$$[\Omega_a, \Omega_b] = 0 \qquad a, b = \overline{1, 4} \tag{17}$$

$$[H, \Omega_a] = 0 \qquad a = \overline{1, 4}. \tag{18}$$

The charges $\Omega_a$ are decomposed in the form

$$\Omega_a = \sum_{r \geqslant 0} \overset{(r)}{\Omega}_a \qquad \mathrm{res}(\overset{(r)}{\Omega}_a) = r \tag{19}$$

and satisfy the boundary conditions

$$\begin{aligned} &\overset{(0)}{\Omega}_a = G_\alpha \delta_{ab} Q^{\alpha b} &&\overset{(1)}{\Omega}_a = -\epsilon_{abcd} P_{\alpha b} \lambda^{\alpha cd} + \cdots \\ &\overset{(2)}{\Omega}_a = \pi_{\alpha ab} \lambda^{\alpha b} + \cdots &&\overset{(3)}{\Omega}_a = -\pi_{\alpha a} \lambda^\alpha + \cdots. \end{aligned} \tag{20}$$

For the Hamiltonian we write

$$H = \sum_{r \geqslant 0} \overset{(r)}{H} \qquad \mathrm{res}(\overset{(r)}{H}) = r \qquad \overset{(0)}{H} \equiv H_c(q, p). \tag{21}$$

As we have already mentioned, Yang–Mills theory is a first-rank theory. One can check that the form of the Hamiltonian (21) is a generalization of the sp(3) case given in [11]:

$$H = H_c + V_\alpha^\beta [P_{\beta a} Q^{\alpha a} + \pi_{\beta ab} \lambda^{\alpha ab} + \pi_{\beta a} \lambda^{\alpha a} + \pi_\beta \lambda^\alpha]. \tag{22}$$

Relation (22) gives all non-minimal terms that, for the sp(4) case, could be expected. In the next section we shall identify some of these terms with the non-minimal terms usually encountered in the standard BRST quantization of the Yang–Mills fields.

## 3. An example of Yang–Mills theory

When Yang–Mills theory is defined on a group with non-vanishing structure functions, we have the case of a non-Abelian model. We consider the case of a $d$-dimensional group, for which the Hamiltonian is given by the relation

$$H_0(x) = \tfrac{1}{2} F^u_{ij} F^{ij}_u - \tfrac{1}{2} p_{iu} p^{iu} \qquad i, j = 1, 2, 3, u = 1, \dots, d. \tag{23}$$

The constraints of this model are

$$G_\alpha \equiv \{G_u \equiv \overset{(2)}{\Phi}_u, G_{d+u} \equiv \overset{(1)}{\Phi}_u; u = 1, \dots, d\} \qquad \alpha = 1, \dots, 2d. \tag{24}$$

where

$$\overset{(2)}{\Phi}_u(x) \equiv -\partial^i p_{iu} + f^t_{uv} A^{iv} p_{it} = 0 \qquad \overset{(1)}{\Phi}_u(x) \equiv p_{ou} = 0. \tag{25}$$

For the system characterized by constraints (25), the density of the canonical Hamiltonian has the form

$$H_c = H_0 + A^{0u} \cdot \overset{(2)}{\Phi}_u. \tag{26}$$

Using notation (24), the constraints can be joined in the $2d \times 2d$ matrix

$$W_{\alpha\beta} = ([G_\alpha, G_\beta]) = \begin{pmatrix} f^t_{uv} G_t & 0 \\ 0 & 0 \end{pmatrix}.$$

We can also write

$$[H_c, G_\alpha] = \begin{cases} A^{0u} f^v_{u\alpha} G_v & \alpha \leqslant d \\ \delta^{u+d}_\alpha G_u & \alpha > d. \end{cases}$$

From (1) we obtain the identifications

$$C^\gamma_{\alpha\beta} \equiv \begin{cases} 0 & \alpha \text{ or } \beta \text{ or } \gamma > d \\ f^\gamma_{\alpha\beta} & \alpha, \beta, \gamma \leqslant d \end{cases} \tag{27}$$

$$V^\beta_\alpha \equiv \begin{pmatrix} A^{0u} f^t_{uv} & 0 \\ \delta^v_u & 0 \end{pmatrix}. \tag{28}$$

Let us decompose the generators of the sp(4) extended phase space as

$$P_{\alpha a} \equiv \{\overset{(2)}{P}_{ua}, \overset{(1)}{P}_{ua}\} \qquad \pi_{\alpha ab} \equiv \{\overset{(2)}{\pi}_{uab}, \overset{(1)}{\pi}_{uab}\} \qquad \pi_{\alpha a} \equiv \{\overset{(2)}{\pi}_{ua}, \overset{(1)}{\pi}_{ua}\} \qquad \pi_\alpha \equiv \{\overset{(2)}{\pi}_u, \overset{(1)}{\pi}_u\}$$

$$Q^{\alpha a} \equiv \{\overset{(2)}{Q}{}^{ua}, \overset{(1)}{Q}{}^{ua}\} \qquad \lambda^{\alpha ab} \equiv \{\overset{(2)}{\lambda}{}^{uab}, \overset{(1)}{\lambda}{}^{uab}\} \qquad \lambda^{\alpha a} \equiv \{\overset{(2)}{\lambda}{}^{ua}, \overset{(1)}{\lambda}{}^{ua}\} \qquad \lambda^\alpha \equiv \{\overset{(2)}{\lambda}{}^u, \overset{(1)}{\lambda}{}^u\}.$$

With (27) and (28), the sp(4) BRST Hamiltonian of the Yang–Mills fields comes to

$$\begin{aligned} H = H_c + \overset{(2)}{P}_{ta} (A^{0u} f^t_{uv} \overset{(2)}{Q}{}^{va} + \overset{(1)}{Q}{}^{ta}) + \overset{(2)}{\pi}_{tab} (A^{0u} f^t_{uv} \overset{(2)}{\lambda}{}^{vab} + \overset{(1)}{\lambda}{}^{tab}) \\ + \overset{(2)}{\pi}_{ta} (A^{0u} f^t_{uv} \overset{(2)}{\lambda}{}^{va} + \overset{(1)}{\lambda}{}^{ta}) + \overset{(2)}{\pi}_t (A^{0u} f^t_{uv} \overset{(2)}{\lambda}{}^v + \overset{(1)}{\lambda}{}^t). \end{aligned} \tag{29}$$

## 4. Comparison to the standard approach

Let us compare the sp(4) Yang–Mills Hamiltonian (29) to that given by standard BRST theory.

We note that, in standard BRST theory, the quantization of an irreducible system is achieved in a phase space [13]:

$$M \equiv \{A^{\mu u}, p_{\mu u}, Q^{\alpha}, P_{\alpha}\}.$$

Using the notation $Q^{\alpha} \equiv \{\overset{(2)}{Q}{}^{u}, \overset{(1)}{Q}{}^{u}\}$ and $P_{\alpha} \equiv \{\overset{(2)}{P}_{u}, \overset{(1)}{P}_{u}\}$, the BRST-invariant Hamiltonian can be written in this case as

$$H = H_c + \overset{(2)}{P}_{u}\,(A^{0t} f^{u}_{tv}\,\overset{(2)}{Q}{}^{v} + \overset{(1)}{Q}{}^{u})\ . \tag{30}$$

The minimal BRST charge for Yang–Mills theory has the standard form

$$\Omega_{\min} = G_{\alpha} Q^{\alpha} + \tfrac{1}{2} f^{t}_{uv}\,\overset{(2)}{P}_{t}\overset{(2)}{Q}{}^{v}\overset{(2)}{Q}{}^{u}\ .$$

If we observe (27) and (28), the previous relations become

$$H = H_c + V^{\beta}_{\alpha} P_{\beta} Q^{\alpha} \tag{31}$$

$$\Omega_{\min} = G_{\alpha} Q^{\alpha} + \tfrac{1}{2} C^{\gamma}_{\alpha\beta} P_{\gamma} Q^{\alpha} Q^{\beta}. \tag{32}$$

In the standard procedure, one adds to the space $M$ a *non-minimal sector* generated by the pairs $\{E^{\alpha}, P_{E\alpha}\}$ and $\{F^{\alpha}, P_{F\alpha}\}$. It allows one to give to the Hamiltonian (31) the form

$$H' \equiv H + [K, \Omega].$$

An adequate choice of the fermion $K(E, F, P_E, P_F)$ offers a gauge-fixed Hamiltonian $H'$. The whole BRST charge can be written as [13]

$$\begin{aligned}\Omega &= \Omega_{\min} + \Omega_{\text{non-min}} \\ &= G_{\alpha} Q^{\alpha} + \tfrac{1}{2} C^{\gamma}_{\alpha\beta} P_{\gamma} Q^{\alpha} Q^{\beta} + P_{E\alpha} F^{\alpha}.\end{aligned} \tag{33}$$

Let us accommodate the previous expressions to the notation used in section 3 for non-Abelian Yang–Mills theory. We identify

$$\begin{aligned} &Q^{\alpha} \equiv Q^{\alpha 1} && P_{\alpha} \equiv P_{\alpha 1} \\ &E^{\alpha} \equiv Q^{\alpha 2} && P_{E\alpha} \equiv P_{\alpha 2} \\ &F^{\alpha} \equiv \lambda^{\alpha 34} && P_{F\alpha} \equiv \pi_{\alpha 34}. \end{aligned} \tag{34}$$

By simply looking at relations (31) and (33) we can conclude the following.

- The minimal part of the standard BRST charge and the extended Hamiltonian have the form

$$\Omega_{\min} = G_{\alpha} Q^{\alpha 1} + \tfrac{1}{2} C^{\gamma}_{\alpha\beta} P_{\gamma 1} Q^{\alpha 1} Q^{\beta 1} \tag{35}$$

$$H = H_c + V^{\beta}_{\alpha} P_{\beta 1} Q^{\alpha 1}. \tag{36}$$

They contain terms from the 'ground' level, $L^{(0)}$, only and can be interpreted as the first-order approximation of some 'generalized' quantities that can be constructed using the whole ghost spectrum.

- The non-minimal term that appears in (33) is

$$\Omega_{\text{non-min}} = P_{\alpha 2} \lambda^{\alpha 34}. \tag{37}$$

It is constructed with variables from the $L^{(-1)}$ and $L^{(1)}$ levels. It is clear that this represents only one of the possible 'non-minimal terms'.

• In the sp(4) theory, the last terms from the right-hand side of equations (35) and (36) can be put in the most complete forms:

$$\tfrac{1}{2}C^{\gamma}_{\alpha\beta}P_{\gamma a}Q^{\alpha a}Q^{\beta 1} \qquad a = \overline{1,4}$$

and, respectively,

$$V^{\beta}_{\alpha}P_{\beta a}Q^{\alpha a} \qquad a = \overline{1,4}.$$

• Besides the previous terms, the extended Hamiltonian (22) and the corresponding BRST generators contain many other non-minimal terms. The sp(4) theory brings up terms that do not appear in the standard approach. Some of them describe the high-order interactions between the ghost variables.

## 5. Conclusions

This paper presents the sp(4) BRST construction for a particular class of gauge theories. The aim of the paper was to clarify the way in which the implementation of a larger symmetry could influence the structure of the phase space and to explain what non-minimal terms could appear in the Hamiltonian. With our rules, one can keep control of these terms and choose the most adequate ghost structure. The analysis of the standard BRST charge and the Hamiltonian for the Yang–Mills theories shows that the sp(2) construction, with the non-minimal terms offered by it, is enough to obtain a coherent quantization. More sophisticated theories ask for a larger ghost spectrum, a spectrum that could be given by a larger symmetry. So, the conclusions we drove at are as follows: (i) building a generalized BRST symmetry appears as a possibility, the standard and the sp(2) theories proving themselves as the first two stages of this global theory; (ii) by itself, a more extended symmetry asks for a larger ghost spectrum and, so, more non-minimal terms could be employed in the gauge-fixing procedure. The level structure of the variables offers simple rules for constructing these terms.

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